

# Highly Damped Quasinormal Modes of Kerr Black Holes: A Complete Numerical Investigation

Emanuele Berti\*

McDonnell Center for the Space Sciences, Department of Physics,  
Washington University, St. Louis, Missouri 63130, USA. †

Vitor Cardoso‡

Center for Computational Physics, University of Coimbra, P-3004-516 Coimbra, Portugal §

Shijun Yoshida¶

Centro Multidisciplinar de Astrofísica - CENTRA, Departamento de Física,  
Instituto Superior Técnico, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal

(Dated: February 3, 2008)

We compute for the first time very highly damped quasinormal modes of the (rotating) Kerr black hole. Our numerical technique is based on a decoupling of the radial and angular equations, performed using a large-frequency expansion for the angular separation constant  ${}_sA_{lm}$ . This allows us to go much further in overtone number than ever before. We find that the real part of the quasinormal frequencies approaches a non-zero constant value which does *not* depend on the spin  $s$  of the perturbing field and on the angular index  $l$ :  $\omega_R = m\varpi(a)$ . We numerically compute  $\varpi(a)$ . Leading-order corrections to the asymptotic frequency are likely to be  $\sim 1/\omega_I$ . The imaginary part grows without bound, the spacing between consecutive modes being a monotonic function of  $a$ .

PACS numbers: 04.70.-s, 04.30.Nk, 04.70.Bw, 11.25.-w

## I. INTRODUCTION

Black holes (BHs), as many other objects, have characteristic vibration modes, called quasinormal modes (QNMs). The associated complex quasinormal frequencies (QN frequencies) depend only on the BH fundamental parameters: mass, charge and angular momentum. QNMs are excited by BH perturbations (as induced, for example, by infalling matter). The early evolution of a generic perturbation can be described as a superposition of QNMs, and the characteristics of gravitational radiation emitted by BHs are intimately connected to their QNM spectrum. One may in fact infer the BH parameters by observing the gravitational wave signal impinging on the detectors [1]: this makes QNMs highly relevant in the newly born gravitational wave astronomy [2, 3].

Besides this “classical” context, QNMs may find a very important place in the realm of a quantum theory of gravity. General semi-classical arguments suggest [4] that on quantizing the BH area one gets an evenly spaced spec-

trum of the form

$$A_n = 4 \log(k) l_P^2 n ; \quad n = 0, 1, \dots \quad (1)$$

where  $l_P$  is the Planck length, and  $k$  is an integer to be determined. Hod [5] proposed to fix the value of  $k$ , and therefore the area spectrum, by promoting QN frequencies with a very large imaginary part to a special position: they should bridge the gap between classical and quantum transitions. Hod obtained, for the Schwarzschild BH,  $k = 3$ . Following his proposal, further enhanced by the prospect of using similar reasoning in Loop Quantum Gravity to fix the Barbero-Immirzi parameter [6], the interest in highly damped BH QNMs has grown considerably [7]. There is now reason to believe that the connection between QN frequencies and the BH area quantum is not as straightforward as initially suggested. However a relation between classical and quantum BH properties does seem to exist, even in non-asymptotically flat spacetimes [8]. A prerequisite to study this connection is to compute QN frequencies having very large imaginary part. So far this problem has been solved only for a few special geometries: Schwarzschild BHs [9, 10, 11, 12, 13], Reissner-Nordström (RN) BHs [11, 12, 13], the Bañados-Teitelboim-Zanelli BH [14], and the four-dimensional Schwarzschild-anti-de Sitter BH [15].

We must try to include the Kerr geometry in this short catalogue, due to its great importance and simplicity. This is a problem of great relevance for the scientific community, and quite a lot of effort is being invested here. This effort is in direct proportion to the difficulty of the problem. All previous attempts [13, 16, 17, 18]

\*Present address: Groupe de Cosmologie et Gravitation (GReCO), Institut d’Astrophysique de Paris (CNRS), 98<sup>bis</sup> Boulevard Arago, 75014 Paris, France

†Present address: Centro Multidisciplinar de Astrofísica - CENTRA, Departamento de Física, Instituto Superior Técnico, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal

\*Electronic address: berti@iap.fr

‡Electronic address: vcardoso@fisica.ist.utl.pt

¶Electronic address: yoshida@fisica.ist.utl.pt

at probing the asymptotic QNMs of Kerr BHs have been unsuccessful, or at least unsatisfactory. There have been several contradictory “analytical” results, which were either based on incorrect assumptions, or could not probe the highly damped regime [16]. The few numerical results [13, 17, 18] are not decisive either, although they definitely show some trend. A numerical investigation is necessary both as a benchmark and as a guide to analytical approaches. Here we carry out such a numerical study. We improve on previous results by going further in overtone number than ever before, in order to really probe the asymptotic regime. The starting point for our analysis is, as previously [13, 18], Leaver’s continued fraction technique [19] as improved by Nollert [9], with a few appropriate modifications [13]. However, we now decouple the angular and radial equations. We first determine the asymptotic expansion for the angular separation constant, and then replace this asymptotic expansion in the radial equations. This trick spares us the need to solve simultaneously the two equations, which was the major drawback of previous numerical works. A leading-order asymptotic expansion of the separation constant is typically accurate for  $|a\omega| \gtrsim 5$ , where  $a$  is the dimensionless Kerr rotation parameter [20]. In this study we go well beyond this regime (we can usually compute modes up to  $|a\omega| > 50$ , an order of magnitude larger). So we have great confidence that our results really yield “asymptotic” QN frequencies.

We find that our previous results [18] for negative  $m$  and moderately damped QN frequencies were quite close to the true asymptotic behaviour (especially for large values of  $a$ ), while convergence to the asymptotic value was not yet achieved for positive  $m$ . Our improved calculations have been carried out with two independent numerical codes. Our main results are that: (i) The real part of the QN frequencies  $\omega_R$  approaches a non-zero constant value. This value does *not* depend on the spin  $s$  of the perturbing field and on the angular index  $l$ . It only depends on the rotation parameter  $a$ , and is proportional to  $m$ :  $\omega_R = m\varpi(a)$ . We determine  $\varpi(a)$  numerically. A fit of our numerical data by power series in  $1/\omega_I$  and  $\sqrt{1/\omega_I}$  suggests that leading-order corrections to the asymptotic frequency should be of order  $1/\omega_I$ . (ii) The imaginary part  $\omega_I$  grows without bound, the spacing between modes  $\delta\omega_I$  being a monotonically increasing function of  $a$ .

## II. BASIC EQUATIONS

In the Kerr geometry, the condition that a given frequency be a QN frequency can be converted into a statement about continued fractions, which is rather easy to implement numerically. Such a procedure has been explained thoroughly by Leaver [19], so here we shall only recall the basic idea. The perturbation problem reduces to a pair of coupled differential equations: one for the angular part of the perturbations, and the other for the

radial part. In Boyer-Lindquist coordinates, defining  $u = \cos\theta$ , the angular equation reads

$$\begin{aligned} & [(1-u^2)S_{lm,u}]_u + \\ & + \left[ (a\omega u)^2 - 2a\omega su + s + {}_s A_{lm} - \frac{(m+su)^2}{1-u^2} \right] S_{lm} = 0, \end{aligned} \quad (2)$$

and the radial one is

$$\Delta R_{lm,rr} + (s+1)(2r-1)R_{lm,r} + V(r)R_{lm} = 0, \quad (3)$$

where  $\Delta = r^2 - r + a^2$  and

$$\begin{aligned} V(r) = & \left[ (r^2 + a^2)^2 \omega^2 + is(am(2r-1) - \omega(r^2 - a^2)) + \right. \\ & \left. + a^2 m^2 - 2am\omega r \right] \Delta^{-1} + [2is\omega r - a^2 \omega^2 - {}_s A_{lm}]. \end{aligned} \quad (4)$$

The parameter  $s = 0, -1, -2$  for scalar, electromagnetic and gravitational perturbations respectively, and  ${}_s A_{lm}$  is an angular separation constant. In the Schwarzschild limit the angular separation constant can be determined analytically:  ${}_s A_{lm} = l(l+1) - s(s+1)$ .

Boundary conditions for the two equations can be cast as a couple of three-term continued fraction relations [19]. Finding QN frequencies is a two-step procedure: for assigned values of  $s, l, m, a$  and  $\omega$ , first find the angular separation constant  ${}_s A_{lm}(\omega)$  looking for zeros of the *angular* continued fraction; then replace the corresponding eigenvalue into the *radial* continued fraction, and look for its zeros as a function of  $\omega$ . This has been the strategy adopted in earlier works [13, 17, 18, 19], where the first  $\sim 50$  modes were computed. These numerical investigations showed a rich (and perhaps confusing) behavior. For negative  $m$  and large enough  $a$  the first  $\sim 50$  modes display some kind of convergence, and are consistent with our new results. Among modes with positive  $m$ , only those having  $|s| = l = m = 2$  seemed to converge. This convergence was deceiving: positive- $m$  results in [18] were not yet in the asymptotic regime. The “true” asymptotic behavior turns out to be much simpler than the intermediate-damping regime explored in [18].

Since the major numerical difficulty lies in the coupling of the two continued fractions, here we adopt a “trick” to decouple them. We first carry out a careful study of the angular equation, determining the asymptotic form of the separation constant  ${}_s A_{lm}$  for frequencies  $\omega$  with large imaginary part. Then we substitute this expansion in Eqs. (3)-(4). By this trick we reduce the problem to the numerical solution of a *single* three-term recursion relation: it is then possible to probe the asymptotic regime for highly damped modes. In the following section we shall briefly discuss our main analytical and numerical results for the asymptotic expansion of  ${}_s A_{lm}$ .

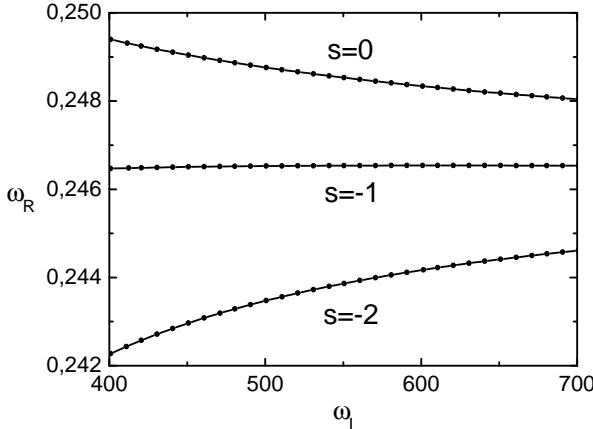


FIG. 1: Real part of high-order QN frequencies for scalar ( $s = 0$ ), electromagnetic ( $s = -1$ ) and gravitational ( $s = -2$ ) perturbations of a Kerr BH with  $a = 0.1$  ( $l = m = 2$ ) as a function of their imaginary part. QN frequencies of different spins converge to the same value. For any kind of perturbation we are already deep in the region of validity of the asymptotic expansion (7), since  $|a\omega| \sim 60$ .

### III. ASYMPTOTIC EXPANSION OF THE ANGULAR SEPARATION CONSTANT

The analytical properties of the angular equation (2) and of its eigenvalues have been studied by many authors [21, 22, 23, 24, 25]. Series expansions of  ${}_s A_{lm}$  for  $|a\omega| \ll 1$  are available, and they agree well with numerical results [25]. On the other hand, the asymptotic behavior for large frequencies has hardly been studied at all. An analytical power-series expansion for large (pure real and pure imaginary) values of  $a\omega$  can be found in Flammer's book [21], but it is limited to the case  $s = 0$ . Flammer's results are in good agreement with the exhaustive numerical work by Oguchi [26], who computed angular eigenvalues for complex values of  $a\omega$  and  $s = 0$ . A review of numerical methods to compute eigenvalues and eigenfunctions for  $s = 0$  can be found in [27]. Quite surprisingly, there are no systematic numerical results for general spin  $s$ , and the few analytical predictions for large values of  $|a\omega|$  do not agree with each other [28]. Here we shall fill this gap, presenting some results for the large- $|a\omega|$  expansion of  ${}_s A_{lm}$ .

A straightforward generalization of Flammer's method can be easily found for general  $s$ . Define a new angular wavefunction  $Z_{lm}(u)$  through [23]

$$S_{lm}(u) = (1 - u^2)^{\frac{m+s}{2}} Z_{lm}(u), \quad (5)$$

and change independent variable by defining  $x = \sqrt{2}cu$ , where  $c^2 = -(a\omega)^2$ . Substitute this in (2) to get:

$$\begin{aligned} & \left[ {}_s A_{lm} - \frac{cx^2}{2} - i\sqrt{2}cx - m(m+1) - \frac{2msx}{\sqrt{2c+x}} \right] Z_{lm} + \\ & + (2c - x^2) Z_{lm,xx} - 2(m+s+1)x Z_{lm,x} = 0. \end{aligned} \quad (6)$$

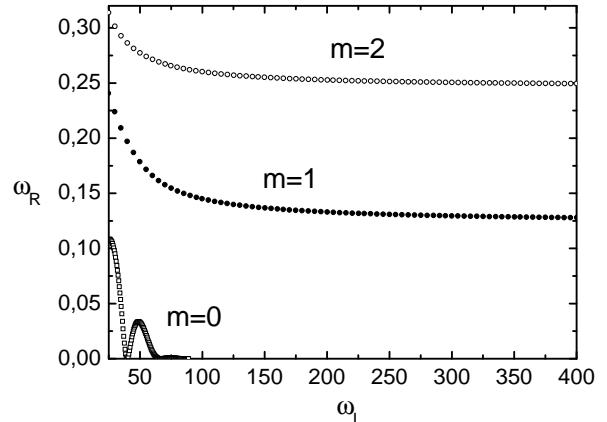


FIG. 2: Scalar QN frequencies of a Kerr BH with  $a = 0.1$ ,  $l = 2$  and different values of  $m$ . The asymptotic value is proportional to  $m$ . Modes with  $m = 0$  oscillate around zero. The amplitude of these oscillations at fixed  $a$  decreases very fast, probably tending to zero as  $\omega_I \rightarrow \infty$ : this is consistent with the behavior shown in Fig. 6 of [18].

When  $c \rightarrow \infty$ , this equation becomes a parabolic cylinder function. The arguments presented in [21, 23, 24] lead to

$${}_s A_{lm} = (2L + 1)c + \mathcal{O}(c^0), \quad c \rightarrow \infty, \quad (7)$$

where  $L$  is the number of zeros of the angular wavefunction inside the domain. One can show that

$$L = \begin{cases} l - |m|, & |m| \geq |s|, \\ l - |s|, & |m| < |s|. \end{cases} \quad (8)$$

Higher order corrections in the asymptotic expansion can be obtained as indicated in [21]. However, we will not need them here. We have verified Eq. (7) numerically, solving Leaver's angular continued fraction for  ${}_s A_{lm}$  as a function of the complex parameter  $a\omega$ . Our numerical results (which will be presented in detail elsewhere) are in excellent agreement with previous work [21, 26, 27] for  $s = 0$ . For any  $s$  they are consistent with equation (7) when  $\omega_R \ll \omega_I$  and  $|a\omega|$  is large.

### IV. NUMERICAL RESULTS

To compute the asymptotic QN frequencies of the Kerr black hole we use a technique similar to that described in [9]. We fix a value of the rotation parameter  $a$ . We first compute QN frequencies for which  $|a\omega| \sim 1$ , so that formula (7) is only marginally valid. This procedure is consistent with our previous intermediate-damping calculations: for example, when we include terms up to order  $|a\omega|^{-2}$  in the asymptotic expansion for  ${}_0 A_{lm}$  provided in [21], our new results for  $a \simeq 0.1$  and  $l = m = 2$  match the results for the scalar case presented in [18] at overtone numbers  $20 \lesssim N \lesssim 30$ . Then we increase the overtone

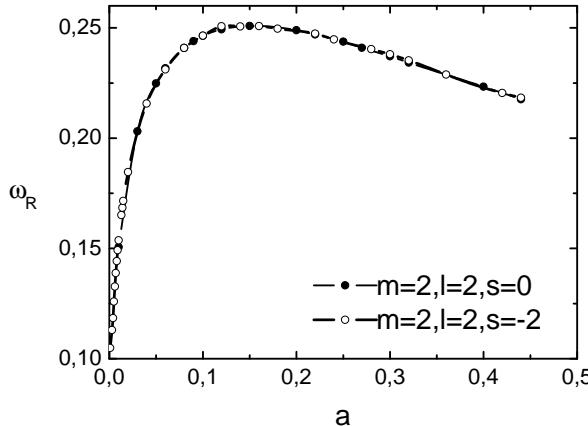


FIG. 3: Asymptotic real part  $\omega_R = 2\varpi(a)$  of the  $l = m = 2$  gravitational and scalar QN frequencies extrapolated from numerical data:  $\omega_R \rightarrow 2\varpi(1/2) \simeq 0.21$  as  $a \rightarrow 1/2$ . The extrapolated frequencies are independent on  $l$  and  $s$ .

index  $N$  (progressively increasing the inversion index of our “decoupled” continued fraction). Finally, we fit our numerical results by functional relations of the form:

$$\omega_R(N) = \omega_R + \omega_R^{(1)}\alpha + \omega_R^{(2)}\alpha^2 + \omega_R^{(3)}\alpha^3, \quad (9)$$

where  $\alpha = 1/\omega_I$  or  $\alpha = \sqrt{1/\omega_I}$ . At variance with the non-rotating case [9], fits in powers of  $1/\omega_I$  perform better, especially for small and large  $a$ . However, both fits break down as  $a \rightarrow 0$ : the values of higher-order fitting coefficients increase in this limit, so that subdominant terms become as important as the leading order, and the extraction of the asymptotic frequency  $\omega_R$  becomes problematic. The numerical behavior of subdominant coefficients supports the expectation (which has not yet been verified analytically) that subdominant corrections are  $a$ -dependent. Therefore one has to be careful to the order in which the limits  $N \rightarrow \infty$ ,  $a \rightarrow 0$  are taken [11, 13, 18]. These observations are consistent with the fact that, in the RN case, the zero-charge limit of the asymptotic QN frequency spectrum does not yield the asymptotic Schwarzschild QN spectrum [11, 12, 13].

We have extracted asymptotic frequencies using two independent numerical codes. For each value of  $a$ , we found that the extrapolated value of  $\omega_R$  is independent of  $s$  (Fig. 1), independent of  $l$  and proportional to  $m$  (Fig. 2):  $\omega_R = m\varpi(a)$ .

We obtained  $\omega_R$  computing QN frequencies both for scalar perturbations ( $s = 0$ ) and for gravitational perturbations ( $s = -2$ ). For definiteness, in both cases we picked  $l = m = 2$ . The agreement between the extrapolated behaviors of  $\omega_R$  as a function of  $a$  is excellent, suggesting that both sets of results are typically reliable with an error  $\lesssim 1\%$ . Our results are also weakly dependent on the number of terms used in the asymptotic expansion of  ${}_sA_{lm}$ : this provides another powerful consistency check.

We have tried to fit the resulting “universal function”, displayed in Fig. 3, by simple polynomials in the BH’s Hawking temperature  $T$  and angular velocity  $\Omega$  (and their inverses). None of these fits reproduces our numbers with satisfactory accuracy. It is quite likely that asymptotic QN frequencies will be given by an implicit formula involving the exponential of the Kerr black hole temperature, as in the RN case [11, 12, 13].

For any  $a$ , the imaginary part  $\omega_I$  grows without bound. Quite surprisingly, the spacing between modes  $\delta\omega_I$  is a monotonically increasing function of  $a$ : it is not simply given by  $2\pi T$ , as recent calculations and previous numerical results suggested [13, 29]. A power fit in  $a$  of our numerical results yields:

$$\delta\omega_I = 1/2 + 0.0438a - 0.0356a^2. \quad (10)$$

## V. CONCLUSIONS

Motivated by recent suggestions of a link between classical black hole oscillations and quantum gravity, we have computed for the first time very highly damped QNMs of the Kerr black hole. Our calculation was made possible by a decoupling of the radial and angular equations, carried out using asymptotic expansions of the angular separation constant  ${}_sA_{lm}$  for  $\omega_r \ll \omega_I$  and  $|a\omega| \gg 1$ . Our results are very weakly dependent on the number of terms used in the asymptotic expansion of  ${}_sA_{lm}$ , and this provides a powerful consistency check. We found that:

(i) The real part of the QN frequencies  $\omega_R$  approaches a non-zero constant value. This value does *not* depend on the spin  $s$  of the perturbing field and on the angular index  $l$ . It only depends on the rotation parameter  $a$ , and is proportional to  $m$ :

$$\omega_R = m\varpi(a). \quad (11)$$

We determined  $\varpi(a)$  numerically (Fig. 3), and showed that it is not given by simple polynomial functions of the black hole temperature  $T$  and angular velocity  $\Omega$  (or their inverses). At fixed  $a$ , a fit of our numerical data by power series in  $1/\omega_I$  and  $\sqrt{1/\omega_I}$  suggests that leading-order corrections to the asymptotic frequency are probably of order  $1/\omega_I$ . (ii) The imaginary part  $\omega_I$  grows without bound, the spacing between modes  $\delta\omega_I$  being a monotonically increasing function of  $a$ .

We wish to stress, once again, that the asymptotic frequency  $\omega_R$  is independent on the spin  $s$  of the perturbing field: this is consistent with results for highly damped QNMs of (charged) RN black holes [11, 12].

By now it is quite clear that the original Hod proposal requires some modification. However, the “universality” of the asymptotic Kerr behaviour we established in this paper is good news. For both charged and rotating black holes the asymptotic QNM frequency  $\omega_R$  depends only on the black hole geometry, not on the perturbing field. If QNMs do indeed play a role in black hole quantization this is an essential prerequisite, and it seems to hold.

## Acknowledgements

We thank all members of the GReCO group, Ted Jacobson, Kostas Kokkotas, Lubos Motl, Andy Neitzke and Cliff Will for their interest in this problem and many useful discussions. This work was partially funded by

Funda o para a Ci ncia e Tecnologia (FCT) – Portugal through project PESO/PRO/2000/4014. VC acknowledges financial support from FCT through PRAXIS XXI programme. SY acknowledges financial support from FCT through project SAPIENS 36280/99.

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